Generalized Artin Schreier Theorem I *

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Abstract

This is an abstract that expands progressively

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Conjecture 1. (Proposition) Let A be a normal (i.e. integrally closed in its total ring of quotient) Baer real ring. Suppose furthermore that T is the total integral closure of A. If T is actually a finitely generated A-module, then A is a real closed $*$ ring and

$$A[\sqrt{-1}] = T$$

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of A, then $A/\mathfrak{p}$ is a real integral domain. There is a $\tilde{\mathfrak{p}} \in \operatorname{Spec} T$ such that $\tilde{\mathfrak{p}} \cap A = \mathfrak{p}$ (see [2] Remark 109). Then we know, by [3] Theorem 1, that $\operatorname{Quot}(T/\tilde{\mathfrak{p}})$ is an algebraically closed field. Since $T/\tilde{\mathfrak{p}}$ is a finite integral extension of $A/\mathfrak{p}$, we can clearly see that $\operatorname{Quot}(T/\tilde{\mathfrak{p}})$ is a finite field extension of $\operatorname{Quot}(A/\mathfrak{p})$ (see [2] Lemma 46 and [1] Proposition 2.1.10). By the classical Artin-Shreier Theorem we then know that $\operatorname{Quot}(A/\mathfrak{p})$ is a real closed field.

Since A is normal, for any minimal prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, $A/\mathfrak{p}$ is integrally closed in its quotient field $\operatorname{Quot}(A/\mathfrak{p})$ (which is a real closed field). And so, by [4] Propositon 2, $A/\mathfrak{p}$ is a real closed integral domain for every minimal prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. By [2] Corollary 102 it follows that A is then real closed.

Define $i := \sqrt{-1_A}$. Then clearly $i \in T$, so by (classical) Artin-Schreier Theorem $\operatorname{Quot}(A/\mathfrak{p})$ is a real closed field and

$$\operatorname{Quot}(A/\mathfrak{p})[i_{\mathfrak{p}}] = \operatorname{Quot}(T/\tilde{\mathfrak{p}})$$

where $i_{\mathfrak{p}} = i \bmod \tilde{\mathfrak{p}}$. □

Example. (Example of a real ring that has all the preconditions above without being Baer and where the result does not follow. Union of $\beta\mathbb{N}$). Define the $X := \beta\mathbb{N} \times \{0,1\}$ with $\{0,1\}$ having the discrete topology. Define also $Y := X/\sim$ where

$$(x,1) \sim (y,0) \Leftrightarrow x, y \in \beta\mathbb{N}\backslash\mathbb{N} \text{ and } x = y$$

with the usual quotient topology. Then clearly both X and Y are Stone spaces with X being extremally disconnected. For brevity, we write the image of any $(x,i) \in X$ in Y also as (x,i). And we define $\psi : X \to Y$ to be the canonical surjection from X to Y.

Now we state a few facts:

- Y is not extremally disconnected because the closure of the open set

  $$\{(x,0) : x \in \mathbb{N}\} \subset Y$$

  is not open in Y.

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- Let K be a real closed field and consider the von Neumann regular rings

$$A_Y := \{f : Y \to K \,:\, f^{-1}(k) \text{ is open for all } k \in K\}$$

and

$$A_X := \{f : X \to K \,:\, f^{-1}(k) \text{ is open for all } k \in K\}$$

Then both A_Y and A_X are von Neumann regular rings (-ADD Reference) with prime spectra Y and X respectively. Because of the surjection ψ, we know that the (canonical) map defined by

$$\phi : A_Y \to A_X \qquad f \mapsto f \circ \psi$$

is injective. It is clear that ϕ is a ring monomorphism.

- We claim that if $f : X \to K$ is in A_X then $g : X \to K$ defined by

$$g(x_1, x_2) := \begin{cases} f(x_1, x_2) & x_2 = 0 \\ 0 & x_2 \neq 0 \end{cases}$$

is in A_Y.

- Define now $e : X \to K$ the following way

$$e(x_1, x_2) := \begin{cases} 1 & x_2 = 0 \\ 0 & x_2 \neq 0 \end{cases}$$

We claim that $e \in A_X$. Let $k \in K$, then if $k \notin \{0, 1\}$ we have $f^{-1}(k) = \emptyset$ which is clearly open in X. For $k \in \{0, 1\}$ we have

$$f^{-1}(0) = \{(x_1, x_2) \,:\, x_1 \in \beta\mathbb{N},\, x_2 = 1\}$$

$$f^{-1}(1) = \{(x_1, x_2) \,:\, x_1 \in \beta\mathbb{N},\, x_2 = 0\}$$

which are also open in X. Thus $e \in A_X$.

- We claim that $A_Y[e] = A_X$
Let $f \in A_X$, then define $g_1 : X \to K$ by

$$g_1(x_1, x_2) := f(x_1, 0)$$

and $g_2 : X \to K$ by

$$g_2(x_1, x_2) := f(x_1, 1)$$

First we claim that g_1 and g_2 are in A_Y, but this is clear since for any

$$g_i(x, 0) = g_i(x, 1) \qquad \forall x \in \beta\mathbb{N}, i = 1, 2$$

We can then easily check that $f = g_1 e + g_2(1 - e)$ and thus conclude that $f \in A_Y[e]$.

Because X is extremally disconnected and Y is not, we know that A_X is Baer and A_Y is not Baer. It is also clear that e is a rational element of A_Y. Now define $e' : X \to K$ the following way

$$e'(x_1, x_2) := \begin{cases} 1 & x_1 \in \mathbb{N} \text{ and } x_2 = 0 \\ 0 & \text{else} \end{cases}$$

Then $e' \in A_X$ because $e'(x, 0) = e'(x, 1)$ for all $x \in \beta\mathbb{N}\backslash\mathbb{N}$ and for any $k \in K$ the set $e'^{-1}(k)$ is open. Finally we also see that e is a rational element of A_Y, since $e' \cdot e \in A_Y\backslash\{0\}$. Since we are dealing with von Neumann regular rings A_X and A_Y we know that these rings are normal (ADD-Tocheck). Because of the preceding Proposition we also know that $A_X[\sqrt{-1}]$ is a total integral closure of A_Y (since A_X is Baer). We also note that e is not in $A_X[\sqrt{-1}]$ and so the result of the proposition above does not hold if we remove the condition that the ring should be Baer.

References

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